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ON THE HOLOMORPHS OF SOLUBLE GROUPS OF FINITE RANK

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1. Discussion of the main results

In [13], Merzljakov shows that the holomorph of a polycyclic group has a faithful representation of finite degree over the integers $\mathbf{Z}$, and in [1] Auslander proves, using Lie-group techniques, that the automorphism group of a polycyclic group is finitely presented. In this paper we give alternative proofs of these two results, proofs that are simpler, but which also yield more information.

We shall say that a group G has finite (Prüfer) rank if there exists an integer r such that every finitely generated subgroup of G can be generated by r elements. Let π be any set of primes. Denote by $\mathfrak{P}_\pi$ the class of all groups G with a series of finite length whose factors are either free abelian of finite rank or abelian π-groups of finite rank or finite. Clearly subgroups and quotient groups of $\mathfrak{P}_\pi$-groups are $\mathfrak{P}_\pi$-groups. Also every torsion-free abelian $\mathfrak{P}_\pi$-group contains a free abelian subgroup of finite rank modulo which the group is an abelian π-group of finite rank. $\mathfrak{P}_\emptyset$ is simply the class of polycyclic-by-finite groups, and if $\emptyset'$ denotes the set of all primes, then $\mathfrak{P}_{\emptyset'}$ is the class of soluble-by-finite groups of finite rank, a class denoted by $\mathfrak{S}\mathfrak{F}$ by a number of authors, notably Robinson [15]. The class of soluble-by-finite minimax groups is

$$\bigcup_{\pi \text{ finite}} \mathfrak{P}_\pi.$$

The above notation is an adaption of that of Hulse [6]. Write R_π for the subring $\mathbf{Z}[1/p;\, p \in \pi]$ of $\mathbf{Q}$.

1.1. Theorem. *If G is a torsion-free-by-finite $\mathfrak{P}_\pi$-group, then the holomorph* Hol G *of G has a faithful representation of finite degree over R_π.*

We obtain immediately the following three corollaries of Theorem 1.1, the first of which, for the special case where G is also nilpotent, is due to Segal [16].

1.2. Corollary. *The holomorph of a finite extension G of a torsion-free soluble group of finite rank has a faithful representation of finite degree over the rationals.*

1.3. Corollary. *If G is a finite extension of a torsion-free soluble minimax group, then there exists a finite set π of primes such that* Hol G *has a faithful representation of finite degree over R_π.*

1.4. Corollary [13]. *The holomorph of a polycyclic-by-finite group has a faithful representation of finite degree over* **Z**.

A direct proof of 1.4 based on our methods would be substantially shorter than Merzljakov's original one. Also we shall not assume the representability of polycyclic groups even over a field: indeed this will be a corollary of our results. Incidentally, proofs that a finite extension of a torsion-free soluble group of finite rank has a faithful representation of finite degree over **Q** are given in [8] and [11]; see also [17, pp. 25–26].

The restriction to torsion-free-by-finite groups in 1.1 is necessary even in the most elementary cases. Suppose that G is the direct product of an infinite cyclic group X and a Prüfer p^∞-group P. If $\Gamma = \operatorname{Aut} G$, $\Xi = \operatorname{Aut} X$, $\Pi = \operatorname{Aut} P$ and $\Sigma = \operatorname{Hom}(X,P)$, then Γ is isomorphic to the split extension $\bar{\Gamma}$ of Σ by $\Xi \times \Pi$, where the latter acts on Σ, in the obvious notation, by

$$\sigma^{(\xi,\pi)}: g \mapsto g^{\xi^{-1}\sigma\pi}.$$

Now Σ is isomorphic to P, so if ρ is any faithful representation of $\bar{\Gamma}$ of finite degree over a field, then $\Sigma\rho$ is absolutely completely reducible [17, 2.2, 1.6] and so $\bar{\Gamma}/C_{\bar{\Gamma}}(\Sigma)$ is finite by [17, 1.12]. Clearly this is false. Hence Γ is isomorphic to no linear group whatever.

Since in the above example G is abelian, the outer automorphism group of G does not have a faithful representation of finite degree over a field. The point of this remark is that the outer automorphism group of a Černikov group does not have such a representation, for example over **C** (see [12] or [7, 3.38]). It is very easy to check that the infinite locally dihedral 2-group D_{2^∞} (see [7, §1.I]) is a Černikov group whose automorphism group has no faithful representation of finite degree over any field (for essentially the same reason that Aut G above does not).

Our second theorem is the following generalization of the result of Auslander quoted above.

1.5. Theorem. *Let G be a polycyclic-by-finite group with automorphism group Γ and suppose that $X, X_1, \ldots, X_r$ is a finite collection of subgroups of* Hol G *such that $X_1, X_2, \ldots, X_r$ are polycyclic-by-finite. Then*

$$C_\Gamma(X) \cap N_\Gamma(X_1) \cap N_\Gamma(X_2) \cap \ldots \cap N_\Gamma(X_r)$$

is finitely presented. In particular, Γ is finitely presented.

In 1.5 above one could replace $C_\Gamma(X)$ by an intersection of centralizers, but since an intersection of centralizers is a centralizer, there is no gain. If X is a polycyclic-by-finite subgroup of the holomorph of the group G in 1.5, then it follows from 1.5 that $N_\Gamma(X)$ and $C_\Gamma(X)$ are both finitely presented. It is an immediate consequence of this that $N_\Gamma(X)/C_\Gamma(X)$ is also finitely presented. The latter group is a sort of relative automorphism group of X. If we no longer assume that X is polycyclic-by-finite, then 1.5 still implies that $C_\Gamma(X)$ is finitely presented, but now $N_\Gamma(X)$ and $N_\Gamma(X)/C_\Gamma(X)$ need not even be finitely generated. This phenomenon can occur moreover even if G is a very straightforward group, such as a free abelian group of rank 2, as 1.6 below shows.

1.6. Example. $L = \mathrm{GL}(2, \mathbf{Z})$ contains a subgroup X such that $X = N_L(X)$ and such that $X/C_L(X)$ is not finitely generated.

While discussing finitely presented linear groups we give a negative answer to a question of Serge Lang. In the final sentence of his book [9] on diophantine geometry Lang asks whether a finitely presented linear group P of characteristic 0 necessarily has an injective specialization into a linear group over the algebraic closure $\mathbf{A}$ of $\mathbf{Q}$. The answer is no, and in a strong sense.

1.7. Example. There exists a finitely presented metabelian linear group P over $\mathbf{C}$ that has no faithful representation of finite degree over $\mathbf{A}$.

The construction of the group P in 1.7 depends on certain large unipotent groups. Hence it might be reasonable to hope that the following has a positive answer: *If P is a finitely presented subgroup of* $\mathrm{GL}(n, \mathbf{C})$ *each of whose elements is diagonalizable, does P have a faithful representation of finite degree over* $\mathbf{A}$ *(and hence over* $\mathbf{Q}$*)? If so, does P even have an injective specialization over* $\mathbf{A}$?

1.8. *Notation and conventions*

Let G be a group, S a subset of G and H a subgroup of G. We use the following notation: G' denotes the derived group of G, $C_G(S)$ the centralizer of S in G, $N_G(S)$ the normalizer of S in G, $\mathrm{Aut}\, G$ the automorphism group of G, $\mathrm{Hol}\, G$ the holomorph of G, $\eta_1(G)$ the Fitting subgroup of G, $\pi(G)$ the set of primes p for which G contains an element of order p, $|S|$ the cardinality of S and $(G : H)$ the index of H in G.

If R is an integral domain and G is a subgroup of $\mathrm{GL}(n,R)$, then G^0 denotes the connected component of G containing 1 in the topology induced on G by the Zariski topology, $u(G)$ the unique maximal unipotent normal subgroup of G (this notation is also used when G more generally is a group of module automorphisms over a commutative ring) and for S a subset of $\mathrm{GL}(n,R)$, $\mathscr{A}_R(S)$ is the intersection of the Zariski-closed subgroups of $\mathrm{GL}(n,R)$ containing S. $\mathrm{Tr}_1(n,R)$ denotes the group of $n \times n$ lower triangular matrices over R with all their diagonal entires 1.

Mappings throughout are written on the right. If G is a group and if $g \in G$ and $\gamma \in \mathrm{Aut}\, G$, then we have usually written g^γ rather than $g\gamma$ for the image of g under γ in order to facilitate calculations in the holomorph.

2. Representations of holomorphs

The key lemma in this section is the following result, which says that the automorphism group of the group G in Theorem 1.1 is substantially smaller than one might at first suspect. Certain infinite factors of G are "almost centralized" by $\mathrm{Aut}\, G$.

2.1. Lemma. *Let G be a finite extension of a soluble group of finite rank such that $\pi(G)$ is finite and let Γ be a group of automorphisms of G. Then for every pair of Γ-invariant subgroups M and N of G with $\eta_1(M) \subseteq N \triangleleft M$ and M/N abelian and torsion-free-by-finite, the group $\Gamma/C_\Gamma(M/N)$ is finite.*

Proof. We use a bar to denote "modulo N", so $\bar{M} = M/N$, etc. There exists a positive integer e such that $\bar{M}^e$ is torsion-free. Since $\bar{M}$ has finite rank, $\bar{M}/\bar{M}^e$ is finite. Clearly $\mathrm{Hom}(\bar{M}/\bar{M}^e, \bar{M}^e)$ is trivial, so

$$C_\Gamma(\bar{M}) = C_\Gamma(\bar{M}/\bar{M}^e) \cap C_\Gamma(\bar{M}^e).$$

Also $\Gamma/C_\Gamma(\bar{M}/\bar{M}^e)$ is finite. Hence it suffices to prove that $\Gamma/C_\Gamma(\bar{M}^e)$ is finite. But this group is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbf{Q})$, where $n = \mathrm{rank}\, \bar{M}^e$, and every periodic subgroup of $\mathrm{GL}(n, \mathbf{Q})$ is finite [17, 9.33]. Thus in fact it suffices to prove that $\Gamma/C_\Gamma(\bar{M}^e)$ is merely periodic.

We may assume that $M = NM^e$, i.e., that $\bar{M}$ is torsion-free. Let $\gamma \in \Gamma$ and set $M_\gamma = \langle M, \gamma \rangle \subseteq \mathrm{Hol}\, G$. Then $M_\gamma \in \mathfrak{P}_{\mathfrak{G}}$ and $\pi(M_\gamma)$ is finite. Hence by a theorem of Mal'cev (cf. [15, 3.25, 9.34]) M_γ is nilpotent-by-abelian-by-finite. Further, if $N_\gamma = \eta_1(M_\gamma)$, then $M \cap N_\gamma \subseteq N$. Consequently, $M \cap NN_\gamma = N$, and $\bar{M}$ and MN_γ/NN_γ are M_γ-isomorphic. Thus if r denotes the index of an abelian normal subgroup of M_γ/N_γ of finite index, then $[\bar{M}^r, \gamma^r] = \langle 1 \rangle$. But $\bar{M}$ is torsion-free abelian, so $[\bar{M}^r, \gamma^r] = [\bar{M}, \gamma^r]^r$ and $[\bar{M}, \gamma^r] = \langle 1 \rangle$. This says that $\gamma^r \in C_\Gamma(\bar{M})$ and consequently that $\Gamma/C_\Gamma(\bar{M})$ is periodic.

If we drop in 2.1 the requirement that $\pi(G)$ is finite and replace $\eta_1(M)$ by the Hirsch–Plotkin radical of M, then the lemma remains true. For the automorphism group of a soluble-by-finite group G to be isomorphic to a linear group, G must necessarily satisfy something akin to 2.1, as the following proposition shows.

2.2. Proposition. *Let G be a soluble-by-finite group and for some commutative Noetherian ring R and finitely generated R-module V suppose that $\Gamma = \mathrm{Aut}\, G$ is isomorphic to a subgroup of $\mathrm{Aut}_R V$. Then there exists a characteristic subgroup M of G of finite index containing $N = \eta_1(G)$ such that $\Gamma/C_\Gamma(M/N)$ is finite.*

Proof. Trivially the centre of G lies in N, so it suffices to find such a subgroup M in the inner automorphism group of G and then take its full inverse image in G. Since unipotent groups over R are nilpotent [17, 13.6] we will have $u(M) \subseteq \eta_1(M)$ and the following lemma completes the proof.

2.3. *Let R be a commutative Noetherian ring, V a finitely generated R-module and G a soluble-by-finite normal subgroup of the subgroup H of* $\mathrm{Aut}_R V$. *Then G contains a subgroup M of finite index that is normal in H such that $H/C_H(M/u(M))$ is finite and $M/u(M)$ is abelian.*

Proof. By standard techniques (see [17, ch. 13, especially 13.2 and 13.3]) we can reduce the problem to the case where H is a subgroup of $\mathrm{GL}(n,F)$ for some positive integer n and some algebraically closed field F.

By the Lie–Kolchin Theorem [17, 5.8], $M = G^0$ is triangularizable. Let $\bar{M}$ denote the Zariski closure of M in $\mathrm{GL}(n,F)$. Then H normalizes $\bar{M}$, and there exists [17, 6.4] a rational representation ρ of $H\bar{M}$ with kernel $u(\bar{M})$. It follows from [17, 7.3, 14.20] and the triangularizability that $\bar{M}\rho$ is an abelian d-group, so by [17, 1.12 (and 7.1)] the group $(H\bar{M})\rho/C_{(H\bar{M})\rho}(\bar{M}\rho)$ is finite. Therefore $H/C_H(\bar{M}/u(\bar{M}))$ is finite. Clearly

$$[M, C_H(\bar{M}/u(\bar{M}))] \subseteq M \cap u(\bar{M}) = u(M).$$

Consequently, $H/C_H(M/u(M))$ is finite. Finally $M/u(M)$ is abelian.

2.4. Remark. In [16], Segal shows that the automorphism group of the restricted wreath product of a free abelian group H of rank 2 by itself is not isomorphic to any linear group. This he does by exhibiting a 2-generator soluble subgroup of this automorphism group that is not nilpotent-by-abelian-by-finite. More generally it follows from 2.2 above that the automorphism group Γ of $W = A \wr H$, where A is any non-trivial group and H is as above, is not isomorphic to any linear group.

For suppose to the contrary. Since the centre of W is trivial, Γ contains a copy of W, and hence [17, 10.21] yields that A is abelian. Thus the base group of W is equal to the Fitting subgroup of W. It is very easy to see that $\Xi = \mathrm{Aut}\,H$ embeds into Γ, and indeed even into $N_\Gamma(H)$. It follows easily now from 2.2 that H contains a characteristic subgroup K of finite index such that $\Xi/C_\Xi(K)$ is finite, a contradiction that proves the point.

2.5 Lemma. *Let G be a group and N a torsion-free nilpotent normal $\mathfrak{P}_\pi$-subgroup of G such that G/N is finitely generated. Denote by $\mathfrak{n}$ the ideal of the group ring $R_\pi G$ generated by the set*

$$N - 1 = \{x - 1 : x \in N\} \subseteq R_\pi G,$$

and suppose that $\mathfrak{a}$ is an ideal of $R_\pi G$ such that $\mathfrak{a}$ contains some positive power $\mathfrak{n}^j$ of $\mathfrak{n}$ and such that $R_\pi G/\mathfrak{a}$ is finitely generated as R_π-module. Then $R_\pi G/\mathfrak{a}^i$ is finitely generated as R_π-module for every positive integer i.

Proof. Write R for R_π and let Z_i denote the i^{th} term of the upper central series of N. Now Z_i/Z_{i-1} is a torsion-free abelian $\mathfrak{P}_\pi$-group, and hence it contains a free abelian subgroup of finite rank modulo which Z_i/Z_{i-1} is an abelian π-group. Hence N contains a finitely generated subgroup K such that $Z_i/Z_{i-1}(Z_i \cap K)$ is a π-group for each i.

Let $g \in N$. We claim that $\langle g\rangle/(\langle g\rangle \cap K)$ is a π-group. For by induction on the class of N we may assume that there exists a positive π-number a with $g^a = zk$ for some $z \in Z_1$ and $k \in K$. There exists a positive π-number b with $z^b \in K$. Since z is central in N we have $g^{ab} = z^b k^b \in K$ and our contention is proved.

Trivially G contains a finitely generated subgroup H with $K \subseteq H$ and $HN = G$. We claim now that $RG = RH + \mathfrak{n}^i$ for every positive integer i. If $g \in G$, then $g = hx$ for some $h \in H$ and $x \in N$. Thus

$$g = h + h(x-1) \in RH + \mathfrak{n}\,,$$

and so $RG = RH + \mathfrak{n}$. There exists a positive π-number c with $x^c \in H$. Now the map of N into $\mathfrak{n}/\mathfrak{n}^2$ given by $y \mapsto (y-1) + \mathfrak{n}^2$ is a homomorphism, so $x^c - 1 \equiv c(x-1)$ (mod $\mathfrak{n}^2$). Hence

$$g - h = h(x-1) \in c^{-1}h(x^c - 1) + \mathfrak{n}^2,$$

and consequently $RG = RH + \mathfrak{n}^2$. Thus for any positive integer i,

$$\mathfrak{n}^i = ((RH \cap \mathfrak{n}) + \mathfrak{n}^2)^i \subseteq RH + \mathfrak{n}^{i+1},$$

and so $RG = RH + \mathfrak{n}^i$ for every $i \geqslant 1$.

Trivially $\mathfrak{n}^{ij} \subseteq \mathfrak{a}^i$, so $RG = RH + \mathfrak{a}^i$ and $RG/\mathfrak{a}^i$ and $RH/(RH \cap \mathfrak{a}^i)$ are R-isomorphic. Since $(RH \cap \mathfrak{a})^i \subseteq \mathfrak{a}^i$, it suffices to prove that $RH/(RH \cap \mathfrak{a})^i$ is finitely generated as R-module. For $R = \mathbf{Z}$ this is a special case of [17, Point 1 on page 22]. Since the proof given there uses only that $\mathbf{Z}$ is a principal ideal domain, the proof of 2.5 in general may be completed in the same way.

2.6. Lemma. *Let G be a subgroup of* $\mathrm{GL}(n, R_\pi)$ *with a unipotent normal subgroup N such that G/N is finitely generated. If Γ is a group of automorphisms of G such that $[G, \Gamma] \subseteq N$, then the subgroup $G\Gamma$ of* $\mathrm{Hol}\,G$ *has a faithful representation of finite degree over R_π such that the image of N is unipotent.*

Proof. Write R for R_π. The identity map on G induces an R-algebra homomorphism of RG into R_n with kernel $\mathfrak{k}$ say. Let $\mathfrak{n}$ denote the ideal of RG generated by the set $N - 1$. Since N is a unipotent normal subgroup of G we have $\mathfrak{n}^n \subseteq \mathfrak{k}$. Thus $(\mathfrak{n} + \mathfrak{k})^n \subseteq \mathfrak{k}$. Let V denote the R-torsion-free quotient of $RG/(\mathfrak{n} + \mathfrak{k})^n$. Then V is an RG-module which, since $RG/\mathfrak{k}$ is R-torsion-free, is faithful as G-module.

N is in fact a $\mathfrak{P}_\pi$-group since some conjugate of it lies in $\mathrm{Tr}_1(n,R)$. To see the latter, notice that [17, 1.21] implies that the split extension $R^{(n)}N$ of the row space of dimension n over R by N is nilpotent. If C_i denotes the intersection of $R^{(n)}$ with the i^{th} term of the upper central series of $R^{(n)}N$, then C_i is an R-submodule and C_i/C_{i-1} is $\mathbf{Z}$-torsion-free [15, 2.25]. It therefore is R-free and so a basis for $R^{(n)}$ can be chosen to run through the C_i. Thus N is a $\mathfrak{P}_\pi$-group. Hence 2.5 implies that V is finitely generated as R-module, and, being R-torsion-free, is therefore a free R-module of finite rank. Clearly $V\mathfrak{n}^n = \{0\}$, so N acts unipotently on V.

Γ acts on RG by $(\Sigma\,\alpha_i g_i)^\gamma = \Sigma\,\alpha_i g_i^\gamma$, where $\alpha_i \in R$, $g_i \in G$ and $\gamma \in \Gamma$. This makes RG into an $R(G\Gamma)$-module. If $g \in G$ and $\gamma \in \Gamma$, then

$$g^\gamma - g = g([g,\gamma] - 1) \in \mathfrak{n}.$$

Hence $\mathfrak{n} + \mathfrak{t}$ is Γ-invariant, from which it follows that V is actually an $R(G\Gamma)$-module. The natural map of RG onto V is one-to-one on G, and clearly $G\Gamma$ acts faithfully on $G \subseteq RG$. Hence V is a faithful $G\Gamma$-module.

2.7. Remark. If in addition to the hypotheses of 2.6 we have that for each $x \in N$ and $\gamma \in \Gamma$ there exists a positive integer r with $[x, {}_r\gamma] = 1$, then $N\Gamma$ acts unipotently on V. For [17, 1.21] applied to the action of Γ on the upper central factors of N yields that $[N, {}_s N\Gamma] = \langle 1 \rangle$ for some positive integer s. Let $\mathfrak{n}_i$ denote the ideal of RG generated by the set $[N, {}_i N\Gamma] - 1$, so in the notation above $\mathfrak{n}_0 = \mathfrak{n}$.

Let $x \in G$, $y \in [N, {}_i N\Gamma]$, $a \in N$ and $\gamma \in \Gamma$. Then

$$x^{(a\gamma - 1)} = x^\gamma a^\gamma - x = x([x,\gamma]-1)a^\gamma + xa([a,\gamma]-1) + x(a-1) \in \mathfrak{n}_0.$$

Thus $(RG)^{(a\gamma - 1)} \subseteq \mathfrak{n}_0$. Also,

$$\begin{aligned}(x(y-1))^{(a\gamma-1)} &= x^\gamma(y^\gamma - 1)a^\gamma - x(y-1)\\ &= x([x,\gamma]-1)(y^\gamma-1)a^\gamma + xy([y,\gamma]-1)a^\gamma\\ &\quad + x(y-1)a([a,\gamma]-1) + x(y-1)(a-1)\\ &\in \mathfrak{n}_{i+1} + \mathfrak{n}_0\mathfrak{n}_i + \mathfrak{n}_i\mathfrak{n}_0 = \mathfrak{n}_{i+1} + \mathfrak{n}_0\mathfrak{n}_i.\end{aligned}$$

Thus

$$\mathfrak{n}_i^{(a\gamma - 1)} \subseteq \mathfrak{n}_{i+1} + \mathfrak{n}_0\mathfrak{n}_i.$$

For any u and v in RG we have

$$(uv)^{(a\gamma-1)} = u^\gamma v^{(a\gamma-1)} + u^{(\gamma-1)}v.$$

An induction argument then yields

$$\mathfrak{n}_i^{(a\gamma-1)^{n(i+1)}} \subseteq \mathfrak{n}_{i+1} + \mathfrak{n}_0^n\mathfrak{n}_i.$$

It follows that

$$(RG)^{(a\gamma-1)^t} \subseteq \mathfrak{n}_0^n,$$

where $t = 1 + \frac{1}{2}ns(s+1)$. (Incidentally in a special case of the above calculation given on [17, p. 23], t is wrongly derived as $1 + sn$, and I am indebted to D. Segal for preventing me from perpetuating this error.) The above implies that $V(a\gamma - 1)^t = \{0\}$. Hence N acts unipotently on V.

2.8. Proof of 1.1. Let G be a torsion-free-by-finite $\mathfrak{P}_\pi$-group and set $R = R_\pi$. G contains characteristic subgroups M and N such that G/M is finite, M/N is free abelian of finite rank and N is torsion-free, nilpotent and equal to the Fitting subgroup of M (couple [15, 5.29.1] with the proof of [15, 3.25]).

Let c be the nilpotency class of N and $\mathfrak{n}$ the augmentation ideal of N in RN. By a theorem of Jennings ([5, 7.1], but see 2.10 below) the subgroup $N \cap (1 + \mathfrak{n}^{c+1})$ of N periodic. But N is torsion-free, so $V = RN/\mathfrak{n}^{c+1}$ is faithful as N-module. By 2.5 it is also finitely generated as R-module. If T denotes the R-torsion submodule of V, then T is finite. Elementary stability-group theory shows that $C_N(V/T)$ is finite and hence trivial. Thus V/T is an RN-module that is faithful as N-module and free of finite rank as R-module.

Choose elements $a_1, \ldots, a_r$ of M that map onto a free basis of M/N and set $M_i = \langle a_1, \ldots, a_{i-1}\rangle N$. Denote by α_i the automorphism of M_i induced by conjugation by a_i. If α_i has finite order, then some positive power of a_i lies in the centre of M_{i+1} and hence in N. This would contradict the torsion-freeness of M/N and consequently α_i has infinite order. Thus for $i = 1, 2, \ldots, r$ the group M_{i+1} is isomorphic to the subgroup $M_i\langle\alpha_i\rangle$ of $\operatorname{Hol} M_i$. Trivially $[M_i, \alpha_i] \subseteq N$ for each i. Repeated use of 2.6 now shows that there exists a faithful representation over R with finite degree of $MC_{\operatorname{Aut} M}(M/N)$. But by 2.1 the index in $\operatorname{Hol} M$ of $MC_{\operatorname{Aut} M}(M/N)$ is finite, so there exists a faithful representation of $\operatorname{Hol} M$ over R with finite degree (cf. the proof of [17, 2.3]).

By a well-known result on stability groups, $G_{\operatorname{Aut} G}(G/M)$ is isomorphic to a subgroup of the direct product of $|G/M|$ copies of $\operatorname{Hol} M$, the isomorphism being given simply by

$$\gamma \mapsto (\gamma|_M \cdot [\gamma, x])_{x \in G/M}.$$

Further $C_{\operatorname{Aut} G}(G/M)$ has finite index in $\operatorname{Aut} G$. Thus we can construct a faithful representation of $\operatorname{Aut} G$ over R of finite degree. The following lemma completes the proof of the theorem. (Merzljakov completes his proof of 1.4 in the same way.)

2.9. *If G is any group, then* $\operatorname{Hol} G$ *is isomorphic to a subgroup of* $\operatorname{Aut}(G \wr (\mathbf{Z}/2\mathbf{Z}))$.

Proof. Let G_i, $i = 1, 2$, be distinct groups isomorphic to G via $g \mapsto g_i$, and let x be the automorphism of $G_1 \times G_2$ given by

$$x : g_1 h_2 \mapsto h_1 g_2, \qquad g, h \in G.$$

Then $W = \langle x\rangle(G_1 \times G_2)$ is isomorphic to $G \wr (\mathbf{Z}/2\mathbf{Z})$. For $\gamma \in \Gamma = \operatorname{Aut} G$ define the map γ^* of W into itself by

$$(x^i g_1 h_2)\gamma^* = x^i (g\gamma)_1 (h\gamma)_2.$$

Then $*$ is an embedding of Γ into $\operatorname{Aut} W$. Let $*$ also denote the canonical map of G_1 into the inner automorphism group of W. Then $* : G_1 \to G_1^*$ is a Γ^*-isomorphism, $G^* \cap \Gamma^* = \langle 1 \rangle$, and the subgroup $G^*\Gamma^*$ of $\operatorname{Aut} W$ is isomorphic to $\operatorname{Hol} G$.

2.10. *Remark on the use of Jennings' Theorem.* In the proof 2.8 of Theorem 1.1 we can avoid quoting Jennings' Theorem in the following way. All we used is the following special case: Let N be a torsion-free nilpotent $\mathfrak{P}_\pi$-group and set $\mathfrak{n}$ equal to the augmentation ideal of N in RN; then there exists a positive integer k with $N \cap (1 + \mathfrak{n}^k) = \langle 1 \rangle$.

This fact can easily be derived from 2.7 as follows (also although all we require is the case $R = R_\pi$, the proof indicated below works for any ring R with an identity). Let d denote the nilpotency class of N and r the sum of the ranks of the upper central factors of N. We show that $k = 1 + c + c^2 + \ldots + c^{r+1}$ suffices, where $c = \frac{1}{2}d(d+1)$.

Let T be any finitely generated subgroup of N. T contains an abelian normal subgroup A such that $A = C_T(A)$. Since centralizers in T are isolated, T/A is torsion-free. Hence T has a normal series $A = A_0 \subset A_1 \subset \ldots \subset A_s = T$ with $s \leqslant r$ (and possibly zero) and with each A_i/A_{i-1} infinite cyclic. Let $A_{i+1} = A_i \langle a_i \rangle$ for each i and let α_i denote the automorphism of A_i induced by conjugation by a_i. If α_i has finite order, then $a_i^{|\alpha_i|} \in C_T(A) = A$. This contradicts the torsion-freeness of T/A. Consequently, A_{i+1} is isomorphic to the subgroup $A_i \langle \alpha_i \rangle$ of $\operatorname{Hol} A_i$.

Let $\mathfrak{a}_i$ denote the augmentation ideal of A_i in RA_i. Clearly $A_0 \cap (1 + \mathfrak{a}_0^2) = \langle 1 \rangle$. Suppose that $A_i \cap (1 + \mathfrak{a}_i^{t_i}) = \langle 1 \rangle$. If we can show that $A_{i+1} \cap (1 + \mathfrak{a}_{i+1}^{t_{i+1}}) = \langle 1 \rangle$, where $t_{i+1} = 1 + ct_i$, then by induction we have $T \cap (1 + \mathfrak{t}^k) = \langle 1 \rangle$, where $\mathfrak{t}$ is the augmentation ideal of T in RT. Since this is for every finitely generated subgroup T of N, we will have shown that $N \cap (1 + \mathfrak{n}^k) = \langle 1 \rangle$.

Set $\mathfrak{n}_j = RA_i([A_{i,j}\, A_{i+1}] - 1)$, so $\mathfrak{n}_0 = \mathfrak{a}_i$ and $\mathfrak{n}_d = \{0\}$. A_{i+1} acts faithfully on $RA_i/\mathfrak{n}_0^{t_i}$ via the action described in the proof of 2.6. Also the computations in 2.7 show that

$$RA_i^{\mathfrak{a}_{i+1}} \subseteq \mathfrak{n}_0, \qquad \mathfrak{n}_j^{\mathfrak{a}_{i+1}} \subseteq \mathfrak{n}_{j+1} + \mathfrak{n}_0 \mathfrak{n}_j.$$

Hence

$$RA_i^{\mathfrak{a}_{i+1}^{(1+ct_i)}} \subseteq \mathfrak{n}_0^{t_i}.$$

Since A_{i+1} acts faithfully on $RA_i/\mathfrak{n}_0^{t_i}$, this shows that

$$A_{i+1} \cap (1 + \mathfrak{a}_{i+1}^{(1+ct_i)}) = \langle 1 \rangle.$$

3. Finitely presented groups

3.1. Lemma. *If z is a non-trivial unipotent element of* $\mathrm{GL}(n, \mathbf{Q})$, *then $\mathcal{A}_{\mathbf{Q}}(z)$ is isomorphic to the additive group of* $\mathbf{Q}$.

This is well known. Let φ_n denote the map of $\mathbf{Q}$ into $\mathrm{Tr}_1(n, \mathbf{Q})$ given by $a\varphi_n = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ \dfrac{a(a-1)\dots(a-i+j-1)}{(i-j)!} & \text{if } i > j. \end{cases}$$

Direct computation shows that $\mathbf{Q}\varphi_n = \mathscr{A}_{\mathbf{Q}}(J_n(1))$, where $J_n(1)$ is the $n \times n$ Jordan matrix with eigenvalue 1 and $n > 1$. Note that $a_{21} = a$, so φ_n is injective with inverse map given by $(a_{ij}) \mapsto a_{21}$ provided $n > 1$.

In general there exists $x \in \mathrm{GL}(n, \mathbf{Q})$ with $z^x = \mathrm{diag}\,(J_{n_1}(1), \dots, J_{n_r}(1))$, where $n_1 > 1$. Thus

$$\mathscr{A}_{\mathbf{Q}}(z) = x\{\mathrm{diag}(a\varphi_{n_1}, \dots, a\varphi_{n_r}) \colon a \in \mathbf{Q}\}\, x^{-1} \cong \mathbf{Q}$$

3.2. Lemma. *If G is a subgroup of $\mathrm{Tr}_1(n, \mathbf{Q})$, then the Zariski closure $\bar{G}$ of G in $\mathrm{GL}(n, \mathbf{Q})$ is equal to the isolator $\hat{G}$ of G in $\mathrm{Tr}_1(n, \mathbf{Q})$.*

Proof. Since $\hat{G}$ is nilpotent, G is subnormal in $\hat{G}$, say $G = G_0 \lhd G_1 \lhd \dots \lhd G_c = \hat{G}$. Trivially each G_i/G_{i-1} is periodic. Suppose that $G_{i-1} \subseteq \bar{G}$. Then $\bar{G} = \mathscr{A}_{\mathbf{Q}}(G_{i-1})$ and so G_i normalizes $\bar{G}$. By [17, 6.4, 6.6] the group $G_i\bar{G}/\bar{G}$ is isomorphic to a unipotent linear group over $\mathbf{Q}$ and hence is torsion-free. Therefore $G_i \subseteq \bar{G}$. This shows that $\hat{G} \subseteq \bar{G}$.

If $G = \langle 1 \rangle$, the result is trivial, so assume otherwise. Thus G contains a non-trivial central element z. Set $Z = \mathscr{A}_{\mathbf{Q}}(z)$. Then Z is a central subgroup of $\bar{G}$ and there exists by [17, 6.4, 6.6], again, a rational representation ρ of $\bar{G}$ with kernel Z onto a unipotent linear group over $\mathbf{Q}$. Clearly $\bar{G}\rho \subseteq \mathscr{A}_{\mathbf{Q}}(G\rho)$ and by induction on the Hirsch number of G we may assume that every non-trivial cyclic subgroup of $\mathscr{A}_{\mathbf{Q}}(G\rho)$ intersects $G\rho$ non-trivially. Hence if $g \in \bar{G}$, there exist $x \in G$, $y \in Z$ and a positive integer r satisfying $g^r = xy$. By 3.1 the group Z is isomorphic to $\mathbf{Q}$, so $y^s \in \langle z \rangle \subseteq G$ for some positive integer s. Further y is central in $\bar{G}$ and hence $g^{rs} = x^s y^s \in G$. But clearly $\bar{G} \subseteq \mathrm{Tr}_1(n, \mathbf{Q})$; consequently $g \in \hat{G}$. We have shown that $\hat{G} = \bar{G}$.

3.3. Corollary. *If G is any unipotent-by-finite subgroup of $\mathrm{GL}(n, \mathbf{Z})$, then G has finite index in $\mathscr{A}_{\mathbf{Z}}(G)$.*

Proof. Clearly $\mathscr{A}_{\mathbf{Z}}(G) = G\mathscr{A}_{\mathbf{Z}}(G^0)$. Also G^0 is unipotent, so $\mathscr{A}_{\mathbf{Z}}(G^0)$ is a finitely generated nilpotent group. Moreover, by 3.2 every non-trivial cyclic subgroup of $\mathscr{A}_{\mathbf{Z}}(G^0)$ intersects G^0 non-trivially. Thus G^0 has finite index in $\mathscr{A}_{\mathbf{Z}}(G^0)$, and consequently G has finite index in $\mathscr{A}_{\mathbf{Z}}(G)$.

3.4. We recall the following facts about finitely presented groups.

(a) *An extension of a finitely presented group by a finitely presented group is finitely presented* ([4], or see [15, 1.43]).

(b) *If H is a subgroup of finite index in a finitely presented group, then H is finitely presented* [10, p. 93, Corollary 2.8].

(c) *If H is a finitely generated, normal subgroup of the finitely presented group G, then G/N is finitely presented* (trivial).

(d) *If G is finitely generated and abelian-by-finite, then G is a finitely presented group.* (E.g. use (a) and (b) above.)

In what follows, normalizers and centralizers with an asterisk subscript are to be taken in $\mathrm{GL}(n,\mathbf{Z})$, so for example $N_*(G)$ denotes $N_{\mathrm{GL}(n,\mathbf{Z})}(G)$. Also, if X/Y is a factor of some group G, then $C_G(X/Y)$ stands for the subgroup

$$\{x \in G\colon Y^x = Y \text{ and } [X,x] \subseteq Y\}.$$

3.5. Lemma. *Let G be any polycyclic-by-finite subgroup of* $\mathrm{GL}(n,\mathbf{Z})$. *Then $N_*(G)$ contains a normal subgroup S containing $C_*(G)$ with $N_*(G)/S$ finitely generated and abelian-by-finite such that S has finite index in some Zariski closed subgroup C of* $\mathrm{GL}(n,\mathbf{Z})$.

Proof. Set $N = u(G)$. Then N is a closed subgroup of G that is normal in $N_*(G)$. By 2.3 there exists a normal subgroup M of $N_*(G)$ such that $N \subseteq M \subseteq G$ and G/M and $N_*(G)/C_{N_*(G)}(M/N)$ are finite. Put $S = C_*(G/N)$. It follows from the theory of stability groups that $(C_*(M/N) \cap C_*(G/M))/S$ is a finitely generated abelian group and trivially $N_*(G)/C_*(G/M)$ is finite. Therefore $N_*(G)/S$ is a finitely generated abelian-by-finite group.

Let $\bar{N}$ denote the Zariski closure of N in $\mathrm{GL}(n,\mathbf{Z})$. By 3.3 the index $(\bar{N}:N)$ is finite. Thus $e = (G\bar{N}:G)$ is finite. Further $G\bar{N}$ is polycyclic-by-finite, and hence $G\bar{N}$ contains only a finite number of subgroups of index e. Therefore $(N_*(G\bar{N}):N_*(G))$ is finite. The subgroup $C = C_*(G\bar{N}/\bar{N})$ of $N_*(G\bar{N})$ is Zariski closed in $\mathrm{GL}(n,\mathbf{Z})$, for example by [17, 5.10], and $D = C \cap N_*(G)$ has finite index in C. There remains only to prove that $S = D$. But clearly $S \subseteq D$ and

$$[G,D] \subseteq G \cap \bar{N} = N,$$

the final equality following from the closedness of N in G. Thus $S = D$, and the lemma is proved.

3.6. Corollary [1]. *The automorphism group Γ of a polycyclic-by-finite group G is finitely presented.*

In fact 3.6 follows from 3.7 below, but it may make the basic structure of the proof of 3.7 clearer if we prove 3.6 separately first.

Proof. By 1.4 there exists for some integer n an embedding of $\mathrm{Hol}\,G$ into $\mathrm{GL}(n,\mathbf{Z})$, and with an abuse of notation we shall regard $\mathrm{Hol}\,G$ as a subgroup of $\mathrm{GL}(n,\mathbf{Z})$. By 3.5 there exists a subgroup S of finite index in a Zariski closed subgroup C of $\mathrm{GL}(n,\mathbf{Z})$ with $C_*(G)\subseteq S\lhd N_*(G)$ and $N_*(G)/S$ finitely generated and abelian-by-finite. By a theorem of Borel and Harish-Chandra (see [3, p. 14]) Zariski closed subgroups of $\mathrm{GL}(n,\mathbf{Z})$, and in particular C and $C_*(G)$ are finitely presented. By 3.4(b) the group S is finitely presented. Hence $N_*(G)$ is finitely presented by 3.4(d) and 3.4(a). Finally $\Gamma\cong N_*(G)/C_*(G)$ is finitely presented by 3.4(c).

3.7. Proof of 1.5. There exists an integer n for which $\mathrm{Hol}\,G$ can be embedded into $\mathrm{GL}(n,\mathbf{Z})$ and again we shall abuse our notation and regard $\mathrm{Hol}\,G$ as a subgroup of $\mathrm{GL}(n,\mathbf{Z})$. Clearly there is no loss of generality in assuming that $G=X_i$ for some i. Put $Y_i=\Gamma\cap GX_i$. Then $GX_i=GY_i$ and $N_\Gamma(X_i)\subseteq N_\Gamma(Y_i)$. Thus we may also assume that for each i there exists j with $Y_i=X_j$.

By 3.5 there exists a normal subgroup S_i of $N_*(X_i)$ containing $C_*(X_i)$ such that $N_*(X_i)/S_i$ is a finitely generated abelian-by-finite group, and such that S_i has finite index in some Zariski closed subgroup C_i of $\mathrm{GL}(n,\mathbf{Z})$. If $x\in X$, then $x=x_Gx_\Gamma$ for some $x_G\in G$ and $x_\Gamma\in\Gamma$. Set $X_G=\{x_G: x\in X\}$ and $X_\Gamma=\{x_\Gamma: x\in X\}$. Put

$$K=C_*(X_G)\cap C_*(X_\Gamma C_*(G)/C_*(G)).$$

Clearly $K\cap\bigcap_iS_i$ has finite index in the closed subgroup $K\cap\bigcap_iC_i$ of $\mathrm{GL}(n,\mathbf{Z})$. Hence 3.4(b) and the Borel–Harish-Chandra theorem imply that $K\cap\bigcap_iS_i$ is finitely presented. Also $(K\cap\bigcap_iN_*(X_i))/(K\cap\bigcap_iS_i)$ is finitely generated and abelian-by-finite, being isomorphic to a subgroup of the direct product of the $N_*(X_i)/S_i$. Thus 3.4(a) and 3.4(d) yield that $K\cap\bigcap_iN_*(X_i)$ is finitely presented.

Let φ denote the natural projection of $N_*(G)$ onto Γ along $C_*(G)$;

$$[X_i, C_*(G)\cap N_*(Y_i)]\subseteq[GY_i, C_*(G)\cap N_*(Y_i)]\subseteq C_*(G)\cap Y_i=\langle 1\rangle.$$

Hence

(1) $\qquad C_*(G)\cap N_*(Y_i)\subseteq C_*(X_i)$.

Since $\ker\varphi=C_*(G)\subseteq K$ and $G=X_i$ for some i, we have

(2) $\qquad \ker\varphi\cap K\cap\bigcap_iN_*(X_i)=\bigcap_iC_*(X_i)$.

Also φ induces the identity map on Γ and hence on each Y_i, so

$$[Y_i,(N_*(Y_i)\cap N_*(G))\varphi]\subseteq([Y_i,N_*(Y_i)]\cap N_*(G))\varphi=Y_i\varphi=Y_i.$$

Thus

(3) $\qquad (N_*(Y_i)\cap N_*(G))\varphi=N_\Gamma(Y_i)$.

Hence if $x\in N_*(X_i)\cap N_*(Y_i)\cap N_*(G)$, then $(x\varphi)x^{-1}\in C_*(G)\cap N_*(Y_i)$ by (3), which is contained in $C_*(X_i)$ by (1). Thus $X_i^{x\varphi}=X_i^x=X_i$, and we have shown that

(4) $\qquad (\bigcap_iN_*(X_i))\varphi=\bigcap_iN_\Gamma(X_i)$.

Let $c \in C_\Gamma(X)$ and $x \in X$. Then

$$1 = [x_G x_\Gamma, c] = [x_G, c]^{x_\Gamma}[x_\Gamma, c],$$

$$[x_\Gamma, c] = [x_G, c]^{-x_\Gamma} \in G \cap \Gamma = \langle 1 \rangle.$$

This shows that

$$C_\Gamma(X) = C_\Gamma(X_G) \cap C_\Gamma(X_\Gamma) \subseteq K \cap \Gamma.$$

Also

$$[X, K \cap \Gamma] \subseteq [X_G X_\Gamma, K \cap \Gamma] \subseteq [X_\Gamma, K \cap \Gamma] \subseteq C_*(G) \cap \Gamma = \langle 1 \rangle.$$

Therefore $K \cap \Gamma = C_\Gamma(X)$. But $C_*(G) \subseteq K$, so

$$K \cap N_*(G) = (K \cap \Gamma)C_*(G),$$

and hence

$$(5) \qquad (K \cap N_*(G))\varphi = C_\Gamma(X).$$

Putting the pieces together we have the following. The map φ induces a homomorphism of $K \cap \bigcap_i N_*(X_i)$, which by (2) has kernel $\bigcap_i C_*(X_i)$, and which by (5) and (4) maps into and hence onto $C_\Gamma(X) \cap \bigcap_i N_\Gamma(X_i)$. Thus we have

$$C_\Gamma(X) \cap \bigcap_i N_\Gamma(X_i) \cong (K \cap \bigcap_i N_*(X_i))/\bigcap_i C_*(X_i).$$

Now $K \cap \bigcap_i N_*(X_i)$ we have shown above to be finitely presented, and $\bigcap_i C_*(X_i)$ is finitely generated since it is closed in $\mathrm{GL}(n,\mathbf{Z})$. Therefore 3.4(c) implies that $C_\Gamma(X) \cap \bigcap_i N_\Gamma(X_i)$ is finitely presented.

3.8. Remark. In the situation in the proof of 3.6 (or 3.7), if H is any Zariski closed subgroup of $N_*(G)$, then H and $\Gamma \cap HC_*(G)$ are both finitely presented. For set $\bar{H} = \mathcal{A}_{\mathbf{Z}}(H)$. Then $C_{\bar{H}}(G\bar{N}/\bar{N})$ is closed in $\mathrm{GL}(n,\mathbf{Z})$ and so is finitely presented. Since $(N_*(G\bar{N}) : N_*(G))$ is finite and $N_*(G) \cap \bar{H} = H$, it follows from 3.4(b) that $C_H(G\bar{N}/\bar{N})$ is finitely presented. But $C_H(G\bar{N}/\bar{N}) = H \cap S$ (recall that $S = C_*(G/N)$) and $H/(H \cap S)$ is isomorphic to a subgroup of $N_*(G)/S$. Therefore H is finitely presented by 3.4(a) and 3.4(d). Finally $C_H(G) = \bar{H} \cap C_*(G)$ is closed in $\mathrm{GL}(n,\mathbf{Z})$ and so is finitely presented. Consequently 3.4(c) implies that $\Gamma \cap HC_*(G) \cong H/C_H(G)$ is finitely presented.

3.9. Proof of 1.6. Clearly it suffices to exhibit a subgroup N of $G = \mathrm{PGL}(2,\mathbf{Z})$ with $N_G(N) = N$ and $N/C_G(N)$ not finitely generated. Now $S = \mathrm{PSL}(2,\mathbf{Z})$ is the free product of a cyclic group $\langle x \rangle$ of order 2 and a cyclic group $\langle y \rangle$ of order 3 (cf. [10, p. 46, Ex. 19(0)]). Use a bar to denote the natural projection of G onto G/S''. Put $N = N_G(\langle S'', y \rangle)$. Since $\bar{G}$ is an extension of a free abelian group by a group of order 12, the only subgroup of N of order 3 is $\langle \bar{y} \rangle$.

Hence we have

$$N_G(N) \subseteq N_G(\langle \bar{y} \rangle) = N,$$

and so N is equal to its normalizer in G.

Since $(G : S')$ is finite, it suffices to show that $S' \cap N$ is not finitely generated modulo its centre. Now S' is a non-cyclic free group and $S'' \subset S' \cap N \subset S'$, so $S' \cap N$ has no centre, and it suffices to prove that the index $(S' : S' \cap N)$ is infinite (cf. [10, Ex. 29, p. 117]). Clearly $\bar{S}' \cap \bar{N} = C_{\bar{S}'}(\bar{y})$. If $g \in S'$ and $r \in \mathbf{Z}^+$ satisfy $\bar{y}\bar{g}^r = \bar{g}^r\bar{y}$, then $[\bar{g}, \bar{y}]^r = 1$ since $\bar{S}'$ is abelian. As it is also torsion-free we have that $g \in N$ and $(S' : S' \cap N)$ is either infinite or one. In the latter case $\langle \bar{y} \rangle$ is characteristic in $\langle \bar{S}', \bar{y} \rangle$, which is normal in $\bar{S}$. Thus $S \subseteq N$ by the definition of N. But then $x \in N$ and $\bar{S} = \langle \bar{x}, \bar{y} \rangle$ has order 6. This contradiction completes the proof of 1.6.

3.10. Proof of 1.7. Let $G = \langle a \rangle \wr \langle b \rangle$ be the restricted wreath product of two infinite cyclic groups. There exists by the theorem of [2] a finitely presented metabelian group P containing G as a subgroup. If T is the torsion subgroup of the derived group P' of P, then clearly $T \cap G = \langle 1 \rangle$. Further, by the main result of [4], P satisfies the maximal condition on normal subgroups, so P/T is also finitely presented. Hence we may assume that P' is torsion-free. Then P has a faithful representation of finite degree over $\mathbf{C}$ by [14].

P has no faithful representation of finite degree over $\mathbf{A}$ since G does not. To see this, suppose that $G \subseteq \mathrm{GL}(n, \mathbf{A})$. By the Lie–Kolchin theorem [17, 5.8], G has a triangular normal subgroup of finite index m, say. Since G is isomorphic to its subgroup $\langle a^m, b^m \rangle$, this implies that $\mathrm{Tr}(n, \mathbf{A})$ contains a subgroup H isomorphic to G. Since H is finitely generated, in fact $H \subseteq \mathrm{Tr}(n, F)$ for some finite extension field F of $\mathbf{Q}$. Then $H' \subseteq \mathrm{Tr}_1(n, F)$, and the latter group is nilpotent of finite rank, while H' is a free abelian group of infinite rank. This contradiction proves the point.

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